n-tree approximation for the largest Lyapunov exponent of a coupled-map lattice

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The *n*-tree approximation scheme, introduced in the context of random directed polymers, is applied here to the computation of the maximum Lyapunov exponent in a coupled-map lattice. We discuss both an exact implementation for small tree depth *n* and a numerical implementation for larger *n*. We find that the phase transition predicted by the mean-field approach shifts towards larger values of the coupling parameter when the depth *n* is increased. We conjecture that the transition eventually disappears. [S1063-651X(97)04310-9]

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I. INTRODUCTION

The development of analytical techniques to determine Lyapunov exponents in extended systems is certainly an important issue in view of the relevant information provided by them. One cannot, in general, expect to find exact solutions, as the problem is already non-trivial in the case of low-dimensional systems. Since most of the results published in the literature about Lyapunov exponents follow from numerical simulations, the development of effective ''perturbative'' techniques is very welcome as they can also provide information about interactions and correlations that are otherwise undetectable.

More numerical results are certainly available for coupled-map lattices (CMLs) since the discreteness of both space and time variables allows simpler and faster simulations. The most common coupling scheme for a CML is

$$x_i^{t+1}=f(y_i^t),$$

$$y_i^t=\varepsilon x_{i-1}^t+(1-2\varepsilon)x_i^t+\varepsilon x_{i+1}^t. \tag{1}$$

The corresponding evolution equation in the tangent space is

$$\xi_i^{t+1}=f'(y_i^t)[\varepsilon\xi_{i-1}^t+(1-2\varepsilon)\xi_i^t+\varepsilon\xi_{i+1}^t], \tag{2}$$

from which one can see that even the computation of the maximum Lyapunov exponent (MLE) in a CML requires the simultaneous consideration of several issues: (i) space-time correlations of the local multipliers $m_i^t=f'(y_i^t)$, (ii) sign fluctuations of the multipliers that induce partial cancellations in the dynamics of the perturbation ξ_i^t, and (iii) correlations in tangent space induced by the spatial coupling.

The third issue is definitely the first to be clarified as it arises already in the presence of positive δ-correlated multipliers. It is precisely this problem that we shall address in the present paper, trying to determine the MLE in the random matrix approximation, i.e., assuming that all multipliers are independent, identically distributed, random processes. This is the standard assumption made in the study of Anderson localization in disordered systems so that we can say that our investigation can be naturally extended to such a case.

An initial attempt to determine the effect of spatial coupling has been made in [1], where the authors performed a mean-field analysis, exploiting the analogy with the free-energy computation in directed polymers growing in random media. There it was found that the spatial coupling induces a shift in the value of the MLE from the quenched average ($\lambda=\langle\ln m_i\rangle$) in the absence of coupling to the annealed average ($\lambda=\ln\langle m_i\rangle$) above a critical coupling value.

In this paper we apply the so-called *n*-tree approximation scheme [2,3] to obtain more refined analytical estimates of the MLE and to test the convergence properties for increasing depth of the tree. The growing evidence that many features of CMLs dynamics are indeed present also in chains of oscillators and in partial differential equations suggests that techniques developed for CMLs can be extended to such systems.

The paper is arranged as follows. In Sec. II we recall the essential lines of the *n*-tree approximation in directed-polymer theory and reformulate the approach in the present context. Section III is devoted to the numerical implementation, while the small coupling limit is investigated in Sec. IV. Finally, in Sec. V we present some remarks about the problem of estimating the MLE in a coupled map lattice and recall the open problems.

II. METHOD

In this section we first recall the *n*-tree approximation in the context of directed polymers, with reference to a (1+1)-dimensional structure. The approach is then explicitly formulated for the determination of the Lyapunov exponent.

Let us consider all directed walks in a square lattice composed of the displacements $\mathbf{j}\to\mathbf{j}+\mathbf{s}$, where $\mathbf{j}=(i,n)$ represents a generic site, while $\mathbf{s}\in\{(-1,1),(0,1),(+1,1)\}$ (see, e.g., Fig. 1), and attribute a random energy $e(\mathbf{j})$ to each site. The statistical problem amounts to computing the partition function $Z_L(0)$ of all L-step walks departing from the origin,

$$Z_L(0)=\sum_w \exp(-\beta E_w), \tag{3}$$

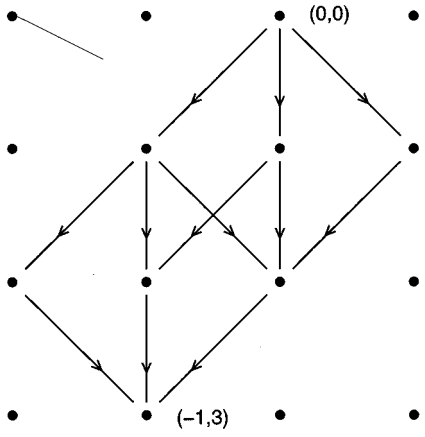

FIG. 1. Schematic diagram of all paths starting from site 0 and arriving at site j in three steps.

where the sum runs over all 3^L paths, E_w is the sum of the energies in the sites visited during the walk, and β is the inverse temperature.

One can write a recursive relation for Z_L,

$$Z_{L+1}(\mathbf{0}) = \exp[-\beta e(\mathbf{0})] \sum_{\mathbf{s}} Z_L(\mathbf{s}), \tag{4}$$

with the initial condition $Z_0(\mathbf{0}) = 1$. The free energy per unit length is nothing but the exponential growth rate of Z with L, $F = -\lim_{L\to\infty} \langle \ln Z_L \rangle / L\beta$, where $\langle\ \rangle$ represents the average over independent disorder realizations.

The main difficulty preventing an exact solution of the above problem is the correlation among the partition functions appearing on the right-hand side of Eq. (4). The n-tree approximation scheme starts from n iterations of Eq. (4) (which automatically accounts for all correlations up to n steps),

$$Z_{L+n}(\mathbf{0}) = \sum_{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n} \exp\left[-\beta \sum_i e(\mathbf{S}_i) \right] Z_L(\mathbf{S}_n), \tag{5}$$

where $\mathbf{S}_i = \Sigma_{m=0,i-1} \mathbf{s}_m$. By embodying all paths ending in the same site j, the above expression can be formally rewritten as

$$Z_{L+n}(\mathbf{0}) = \sum_j M_j Z_L(j), \tag{6}$$

where each multiplier

$$M_j \equiv \sum_{w_j} \exp\left[-\beta \sum_i e(\mathbf{S}_i) \right] \tag{7}$$

is obtained by summing the Boltzmann weights of all paths w_j starting at the origin and ending in j after n steps (see Fig. 1).

It is now convenient to introduce the generating function

$$H_L(x) \equiv \langle \exp\{-e^{-\beta x} Z_L\} \rangle. \tag{8}$$

By inserting Eq. (6) in the formal expression of H_{L+n}, one obtains

$$H_{L+n}(x) = \left\langle \prod_j \exp\{-e^{-\beta x} M_j Z_L(j)\} \right\rangle. \tag{9}$$

The above equation can be turned into a closed recursive relation if the $Z_L(j)$ terms are assumed to be uncorrelated with each other. This is precisely the core of the n-tree approximation scheme, which yields

$$H_{L+n}(x) = \prod_{j=-n}^{n} H_L\left(x - \frac{1}{\beta} \ln M_j \right). \tag{10}$$

The initial condition $H_0 \equiv \langle \exp\{-e^{-\beta x}\} \rangle$ has a sigmoid shape interpolating the two fixed points P_0 ($H=0$) and P_1 ($H=1$) of Eq. (10). Since P_1 is unstable, while P_0 is stable, the ''front,'' where $H(x)$ is sensibly different from either 0 and 1, moves to the right. It can be easily seen that the front velocity is nothing but the free energy of the polymer [2].

The velocity can be determined by approximating the forefront as $H_L(x) \sim 1 - \exp(-\gamma x)$. Substitution of this ansatz into Eq. (10) reveals that the front moves with a velocity that depends on γ,

$$c(\gamma) = \begin{cases} G_n(\gamma), & \gamma \leqslant \gamma_{min} \\ G_n(\gamma_{min}), & \gamma > \gamma_{min}, \end{cases} \tag{11}$$

where

$$G_n(\gamma) = \frac{1}{n\gamma} \ln\left\{ \sum_{j=-n}^{n} \langle M_j^{\gamma/\beta} \rangle \right\} \tag{12}$$

and γ_{min} is the γ value where G_n takes its minimum

$$\left. \frac{dG_n(\gamma)}{d\gamma} \right|_{\gamma_{min}} = 0. \tag{13}$$

The value of γ is implicitly determined by the initial condition: By expanding $H_0(x)$ for large x, one realizes that $\gamma = \beta$. Therefore, if $\beta < \gamma_{min}$, the free energy coincides with the annealed average of the weights M_j, while different γ averages are selected at lower temperatures. This implies the existence of a thermodynamic transition occurring at $\beta_c = \gamma_{min}$.

The analogy between the evolution equation (2) in tangent space and the recursive relation (4) for the partition function suggests that the whole procedure can be extended to the computation of the MLE. In this context, the time t plays the role of the polymer length L and m_i^t can be considered as a quenched noise in a two-dimensional (2D) environment. The growth rate of Z (i.e., the free energy) finally becomes the Lyapunov exponent.

The differences with respect to the previous cases are the presence of the anisotropy factors ε and $1-2\varepsilon$ and the absence of a temperature. The first one is, in principle, only a technical variation, which leads, however, to a strong ''degeneracy'' as we will show in the following. The temperature instead can be removed by setting $\beta = 1$; the role of the relevant control parameter will be played by the coupling strength ε parameter. With the above simple indications, we find that the n-tree approximation Λ_n of the Lyapunov exponent is

$$\Lambda_n = \begin{cases} G_n(\gamma_{min}), & \gamma_{min} < 1 \\ G_n(\gamma = 1), & \gamma_{min} \geqslant 1. \end{cases} \tag{14}$$

The function $G_n(\gamma)$ has the form

$$G_n(\gamma) = \frac{1}{n\gamma} \ln \left\{ \sum_j \left\langle \left(\sum_{w_j} (1-2\varepsilon)^{k(w_j)} \varepsilon^{n-k(w_j)} \times \prod_i m_i(w_j) \right)^{\gamma} \right\rangle \right\}, \tag{15}$$

where we recall that the w_j's are the directed walks on the lattice arriving at site j after n steps (the depth of the tree), $k(w_j)$ is the number of steps not involving a change of position, and $m_i(w_j)$ is the multiplier in the ith time step of the path w_j.

Equation (14) suggests the possible existence of a phase transition upon changing ε: If $\gamma_{min} \geqslant 1$, the Lyapunov exponent is given by the annealed average of the multipliers. The main difficulty in the implementation of this approach is the computation of G_n and its minimization. In the limit $n \to \infty$, the approximation becomes exact: An interesting question concerns the convergence to the asymptotic value. Unfortunately, as n increases, the expression of G_n quickly becomes so complicated that it is practically impossible to handle the analytical expression. For this reason, in the next section we shall address the question from the numerical point of view.

III. NUMERICAL RESULTS

It is true that not only the analytical expression of the MLE becomes very complicated as n increases, but also an ''exact'' numerical implementation is not an easy task. Indeed, the average of disorder implies the computation of several multiple integrals and, even in the simple case of a uniform distribution of multipliers, the presence of a power γ in Eq. (15) makes the integrals immediately undoable. The only case we have found where it is possible to combine an exact solution of the integrals with a powerful numerical analysis is that of a dichotomic distribution. Indeed, a generic multiple integral over K variables becomes a sum over all 2^K combinations of the variables, each properly weighted according to the probability of the two possible values of the multiplier. From Fig. 1, one can see that the number K of integrals to be performed is already equal to 7 for $n=3$ and $j=-1$, which in turn requires summing up 128 different terms. As a result, even in this simple case, it is not possible to go beyond $n=5$.

Since a global rescaling of the multipliers yields a trivial shift of the MLE, we can assume, without loss of generality, that $m_i(t)=\{1,b\}$. Moreover, for the sake of simplicity and in order to maximize the effect of the fluctuations (which are responsible for the deviation of the MLE from the single-map case) we have assumed that 1 and b have the same probability 1/2.

The results for $n=1,\ldots,5$, $b=3$, and small ε values are reported in Fig. 2, where a slow convergence towards the asymptotic value (numerically determined by iterating a chain of 1000 maps) can be observed. In the inset of the same figure, one can also notice that the critical ε value, above which the Lyapunov exponent corresponds to the annealed average, steadily increases, in agreement with the numerical results that do not give any evidence for the existence of the high-temperature phase.

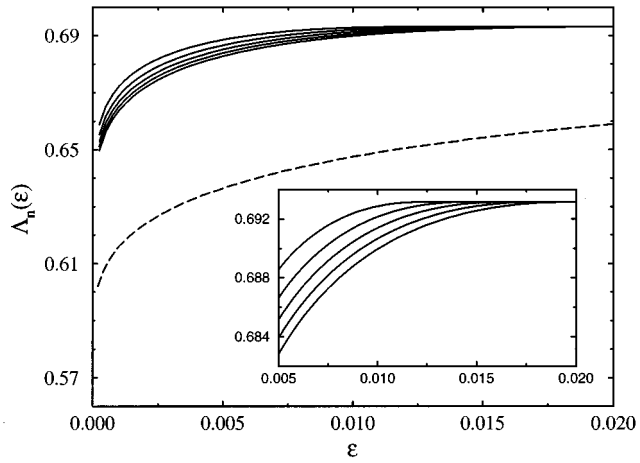

FIG. 2. n-tree approximations of the MLE for $n=1,\ldots,5$ (from top to bottom) compared with the numerical results (dashed curve). These and all the other results in the paper have been obtained for a dichotomic distribution of the local multipliers ($m=\{1,3\}$, with equal probabilities). An enlargement of the region around the supposed phase transition is shown in the inset.

Although an exact implementation of the n-tree scheme is unfeasible already for $n>5$, one can consider it as a numerical algorithm. Indeed, one can imagine to iterate n times a perturbation initially localized in the origin. The amplitude in the site j represents an instance of M_j and the average required in Eq. (15) can be computed by summing over independent realizations of the stochastic process.

Moreover, one can notice that γ plays a similar role to q in the standard multifractal analysis; the only difference is that here, in addition to a local γ average, a linear average over different sites is also required. It is therefore important to understand how γ_{min} behaves for increasing n, i.e., to clarify whether the actual value of the MLE does arise from a specific γ value.

The only drawback of the numerical implementation is the need of a sufficient statistics, a constraint that becomes increasingly important for larger-n values since an accurate determination of M_j^{γ} strongly depends on unprobable large deviations as usual in a multifractal analysis. Notwithstanding this limitation, it has been possible to arrive at $n=50$, much beyond the limits for an exact implementation.

The results for $\varepsilon=0.01$ (reported in Fig. 3) confirm that the approximate values of the MLE approach from above the asymptotic value Λ_∞ (denoted by the horizontal line in the figure). A numerical investigation of the behavior of $\Lambda_n - \Lambda_\infty$ versus n suggests that the convergence is presumably slower than algebraic. The slow variation of Λ_n with n is confirmed by the poor improvement obtained by introducing the refined estimates

$$\tilde{\Lambda}_n = \frac{n\Lambda_n - (n-5)\Lambda_{n-5}}{5} \tag{16}$$

(see the squares in Fig. 3), where the choice of 5 is simply dictated by the spacing of the numerical results. Notice that

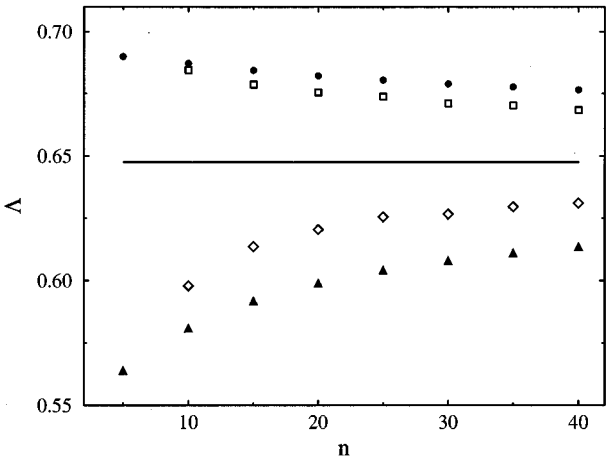

FIG. 3. Results of the n-tree approximation for $\varepsilon=0.01$ (bullets) and refined estimates as from Eq. (16) (squares). Triangles and diamonds refer to the results of the ''naive'' approximation described at the end of Sec. III and to the corresponding refinements, respectively. The horizontal line represents the MLE as determined with the standard numerical procedure.

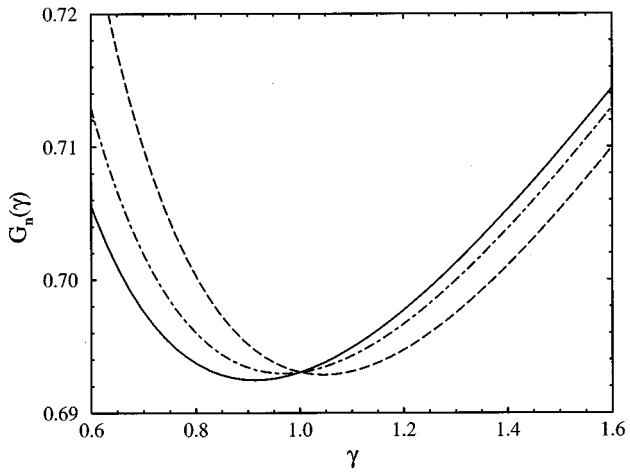

FIG. 4. G_n vs γ for $n=20$ (dashed line), $n=30$ (dot-dashed line), and $n=40$ (full line) obtained from the numerical implementation of the n-tree approximation with $\varepsilon=0.05$.

this procedure is very effective when the main finite-size effect arises from some rapidly decaying initial deviation, as it is the case of the Lyapunov exponent computed with the standard orthonormalization procedure [4].

In addition to allowing the computation of the MLE, the n-tree approximation yields an estimate of the optimal γ_{min} value. By comparing the results for the various depths, one finds that γ_{min} slowly decreases. We conjecture that γ_{min} eventually converges to 0. The conclusion is suggested by the analogy between Eq. (15) and the behavior of the maximal comoving Lyapunov exponent $\lambda_c(v)$ [5], which is defined as the growth rate in the site $i=vt$ of a perturbation initially localized in the origin. In a system with left-right symmetry, $\lambda_c(v)$ reaches its maximal value for $v=0$, where it coincides with the MLE. Therefore, for n large enough, the dominant contribution to $G_n(\gamma)$ is given by the growth rates around the origin, i.e., by their logarithmic average. Accordingly, we expect that γ_{min} will eventually approach 0.

Such a tendency is confirmed for all ε values that we have considered, even well inside the supposed high-temperature phase where $\Lambda_n=G_n(1)$. One such example is illustrated in Fig. 4, where we report $G_n(\gamma)$ for $\varepsilon=0.05$ and different depths n. Notice that all $G_n(\gamma)$ curves take the same value for $\gamma=1$; this is a consequence of the very definition of $G_n(\gamma)$: Independently of n, $G_n(1)$ coincides with the annealed average. For relatively small values of n, the minimum of G_n is attained for $\gamma_{min}>1$ and the correct estimate of the Lyapunov exponent is given by the annealed average. However, upon increasing n, γ_{min} steadily decreases until $\gamma_{min}<1$ (in the case illustrated in Fig. 4, this happens for $20<n<30$). The conjectured convergence of γ_{min} to zero implies the eventual disappearance of the phase transition.

Additional light can be shed on the n-tree approximation scheme shed by comparing it with a similar, though entirely heuristic, approach. The structure of $G_n(\gamma)$ requires iterating an initially localized perturbation: It is therefore quite natural to consider a different initial condition, uniformly spread over $2n$ sites, and to iterate it for n steps by assuming periodic boundary conditions. At variance with the scheme understood in Eq. (15), all sites are now statistically equivalent, so that we can estimate the MLE directly from their growth rate (or, better, from the average growth rate, to reduce statistical fluctuations). The performance of this empirical approach can be judged from the results reported in Fig. 3, where one can see that the convergence is now from below. Moreover, while the direct estimates are worse than the corresponding values obtained from the n-tree scheme, the opposite is true for the improved estimates. We can interpret this result as an indication that the actual value of the MLE in an extended system follows from a delicate balance of several processes. A more effective reduction of finite-size effects can be presumably accomplished by introducing a still different definition of finite-time finite-space Lyapunov exponent. Whether such a definition that applies to generic models exists is not obvious at all.

IV. COUPLING SENSITIVITY

One case that is worth investigating with the aid of the n-tree approximation is the small-coupling limit. In particular, it is interesting to study the scaling behavior of the MLE for decreasing ε. This problem already has been considered in Ref. [1], with the help of a mean-field approach. Here we discuss the improvements arising from the implementation of the n-tree scheme.

In this section, ε will be always so small that even in the lowest approximation ($n=1$) $\gamma_{min}<1$ and the MLE is given by the low-temperature expression. In the one-tree approximation, the Lyapunov exponent is given by $\Lambda(\varepsilon)=G_1(\gamma_{min}(\varepsilon))$, where G_1 follows from Eq. (15) with $n=1$,

$$G_1(\gamma)=\frac{1}{\gamma}\ln\langle m^{\gamma}\rangle+\frac{1}{\gamma}\ln\{(1-2\varepsilon)^{\gamma}+2\varepsilon^{\gamma}\}. \tag{17}$$

For small ε, the above expression simplifies to

$$G_1(\gamma)\cong\frac{1}{\gamma}\ln\langle m^{\gamma}\rangle+\frac{1}{\gamma}\ln(1+2\varepsilon^{\gamma}). \tag{18}$$

Since for $\varepsilon \to 0$ one must recover the value of the uncoupled case, $\gamma_{min}(\varepsilon)$ must go to zero in such a way that also $\varepsilon^{\gamma_{min}(\varepsilon)} \to 0$. Accordingly, one can further expand Eq. (17), obtaining

$$G_1(\gamma) \cong \langle \ln m \rangle + \frac{\Gamma_2}{2}\gamma + 2\frac{\varepsilon^\gamma}{\gamma}, \tag{19}$$

where $\Gamma_2 = \langle (\ln m)^2 \rangle - \langle \ln m \rangle^2$ is the variance of the local expansion rate. With some algebra, it is possible to show that the value of γ_{min} minimizing G_1 is given by

$$\gamma_{min}(\varepsilon) = 2\frac{\ln \sigma}{\sigma}, \tag{20}$$

where $\sigma \equiv |\ln \varepsilon|$ diverges for $\varepsilon \to 0$. On the one hand, formula (20) confirms the correctness of the ansatz $\gamma_{min}(\varepsilon) \to 0$ and that $\varepsilon^\gamma = 1/\sigma$ can be really considered a small parameter. On the other hand, we notice that the convergence to zero is extremely slow.

By substituting Eq. (20) into the expression for G_1 and retaining only the leading terms in ε, we obtain

$$\Lambda_1(\sigma) = \Lambda_0 + \Gamma_2 \frac{\ln \sigma}{\sigma} + \frac{1}{\sigma \ln \sigma}, \tag{21}$$

where $\Lambda_0 = \langle \ln m \rangle$ is the Lyapunov exponent of the uncoupled problem, the second term is responsible for the leading correction, and the third smaller contribution is reported for the sake of completeness.

The above result, already derived in [1], has been recalled here because it allows introducing all the key steps that are necessary to deal with higher-order approximations. In the following we illustrate the case $n=2$ with some detail and mention the further adjustments expected for larger depths of the tree. The complete expression for $G_n(\gamma)$ is already rather complicated for $n=2$,

$$G_2(\gamma) = \frac{1}{2\gamma}\ln\{f_1(\gamma)\} + \frac{1}{2\gamma}\ln\{2\varepsilon^{2\gamma} f_1(\gamma) + 2\varepsilon^\gamma (1-2\varepsilon)^\gamma f_2(\gamma) + f_3(\gamma)\}, \tag{22}$$

where

$$f_1(\gamma) = \langle m^\gamma \rangle,$$

$$f_2(\gamma) = \langle (m_1 + m_2)^\gamma \rangle,$$

$$f_3(\gamma) = \langle (\varepsilon^2 m_1 + (1-2\varepsilon)^2 m_2 + \varepsilon^2 m_3)^\gamma \rangle,$$

while m, m_1, m_2, m_3 represent the multipliers to be averaged.

With the aid of the same approximations made for $n=1$, the above expression simplifies to

$$G_2(\gamma) \cong \langle \ln m \rangle + \frac{\Gamma_2}{2}\gamma + \frac{\varepsilon^\gamma}{\gamma}, \tag{23}$$

which in turn gives the expression for the Lyapunov exponent

$$\Lambda_2(\sigma) = \Lambda_0 + \Gamma_2 \frac{\ln \sigma}{\sigma} + \frac{1}{2}\frac{1}{\sigma \ln \sigma}. \tag{24}$$

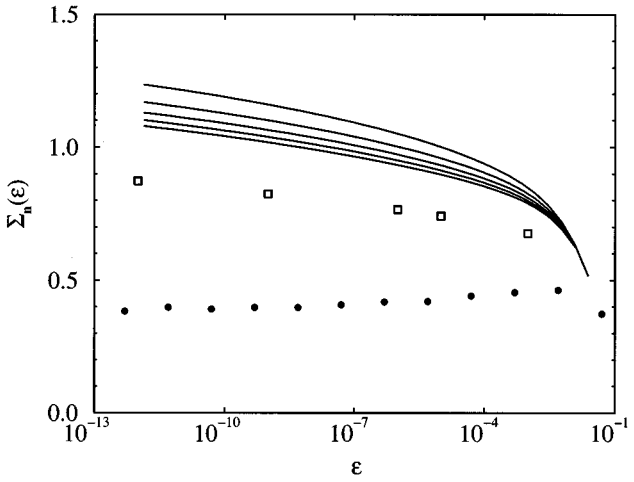

FIG. 5. Σ_n [see Eq. (25)] vs ε for different values of n. The solid lines correspond to $n=1,\ldots,5$ (from top to bottom). Squares denote the results of the $n=40$ approximation, while bullets correspond to the outcome of direct numerical simulations.

By comparing Eqs. (24) and (21), one can see that they differ only in the last term of the right-hand side, which is smaller by a factor 2 in the two-tree approximation. The same is true (apart from the coefficient of the third term) for larger values of n. Accordingly, this analysis seems to indicate that the MLE grows as $\ln|\ln\varepsilon|/\ln\varepsilon$, independently of the depth n. These conclusions are indeed confirmed by the numerical implementation of the n-tree scheme for several depths. This can be noticed in Fig. 5, where the behavior of

$$\Sigma_n \equiv (\Lambda_n - \Lambda_0)|\ln\varepsilon| \tag{25}$$

versus $\ln\varepsilon$ is reported. The slow growth of Σ_n is consistent with the expected $\ln|\ln\varepsilon|$ behavior. However, direct numerical simulations (full dots in Fig. 5) suggest that the MLE grows as $1/\ln\varepsilon$, with no doubly logarithmic correction. In our opinion, the apparent contradiction can be solved by noticing that the determination of the correct scaling behavior requires, in principle, one to take first the limit $n \to \infty$ (to estimate correctly the MLE) and then the limit $\varepsilon \to 0$. In the above analysis, we have instead exchanged the two limits. Since the determination of $G_n(\gamma)$ requires estimating an exponentially growing (with n) number of contributions, it is reasonable to conjecture that increasingly small-ε values must be reached before the leading term really overtakes the others. It is therefore conceivable that, before this asymptotic regime sets in, a different scaling region appears that becomes wider and wider for increasing n. The several limits involved in this process (we must not forget the role of γ) make a rigorous confirmation of this conjecture a rather delicate matter. Here we limit ourselves to recall that in two coupled 1D and 2D maps a purely $1/\ln\varepsilon$ behavior has been proved to arise [6]. However, we cannot exclude that in the present case the scaling ε region is so small that it is has not been reached by our numerical simulations; it will be possible to give a definite answer only by developing a more effective perturbative technique.

Finally, let us notice that the naive approach outlined in Sec. III falls short of identifying the leading $1/|\ln\varepsilon|$ depen-

dence, predicting only linear corrections in ε. We suspect that this failure is due to the lack of an infinite time limit in the corresponding definition of the Lyapunov exponent.

V. CONCLUSIONS AND PERSPECTIVES

In this paper we have implemented the n-tree approximation for the computation of the MLE. The results have revealed a slow convergence towards the asymptotic value. We believe that more than being an intrinsic limitation of the method, this is an indication of the complexity of the problem. This feeling is indeed confirmed by the analogy with the Kardar-Parisi-Zhang (KPZ) equation. As already shown in [7], the logarithm $h=\ln\xi$ of the perturbation approximately satisfies the KPZ equation

$$h_t=\varepsilon h_x x+\varepsilon(h_x)^2+\eta(t), \tag{26}$$

where the subscripts denote derivatives with respect to either time (t) or space (x) variables (assumed now to be continuous) and $\eta(t)$ is a noiselike term corresponding to the logarithm of the local multiplier. Accordingly, the average value of η is nothing but the single-map Lyapunov exponent, while the MLE of the lattice is obtained by adding the average of the nonlinear term. Such a correction can be easily estimated by recalling that in one dimension, the probability distribution of h is exactly the same as for the linear Edwards-Wilkinson model, obtained by neglecting the nonlinearity [8]. The latter is the distribution of the standard Brownian motion, i.e., the product of independent Gaussians for the spatial derivatives h_x [8]. Since the variance of each Gaussian is inversely proportional to the coefficient of the Laplacian, it is seen that $\varepsilon\langle h_x^2\rangle$ is independent of ε. Therefore, the KPZ equation provides for any coupling strength the same annealed average value, i.e., it even fails to find the low-temperature phase. This result indicates that one must go beyond the KPZ approximation of the tangent dynamics (adding higher-order derivatives and further nonlinear terms) if the ε dependence of the MLE is to be recovered.

The identification of effective schemes to fasten the convergence of finite-size estimates of the MLE remains an open problem. Possible routes to be explored in the future are represented by corrections to the KPZ equation or by suitable modifications of the n-tree approximation.

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